The Lamé Solution: An Analytical Treatment of Stresses and Displacements in a Thick-Walled Elastic Cylinder under Internal Pressure

Introduction and Foundational Concepts

1.1 The Engineering Significance of Thick-Walled Cylinders

Cylindrical structures are ubiquitous in engineering, serving as conduits and containers for fluids and gases under pressure. While many applications, such as low-pressure piping and storage tanks, can be adequately analyzed using simplified models, a critical class of components operates under conditions of such high internal pressure that these approximations become invalid and potentially dangerous. These components are classified as thick-walled cylinders, and their analysis forms a cornerstone of advanced solid mechanics.

Prominent examples of thick-walled cylinders include high-pressure hydraulic cylinders, gun barrels designed to withstand intense firing pressures, pipelines for deep-sea oil and gas transport, and critical pressure vessels within chemical and nuclear power plants.[1] In these applications, the wall thickness is a significant fraction of the component's overall radius. The fundamental challenge this geometry presents is that the stresses induced by internal pressure are not uniformly distributed across the wall thickness.[4] The circumferential (or hoop) stress, which resists the bursting of the cylinder, and the radial stress, which acts through the thickness of the wall, both vary significantly from the inner surface to the outer surface.[1] Neglecting this variation, as is done in thin-walled cylinder theory, leads to a gross underestimation of the maximum stresses, which invariably occur at the inner bore of the cylinder. A rigorous analytical solution is therefore essential for the safe and efficient design of these high-performance components. This report provides a definitive, first-principles derivation and analysis of this classical solution, known as the Lamé problem.

1.2 The State of Stress in Cylindrical Coordinates

To accurately describe the internal forces within the cylinder wall, it is most natural to use a cylindrical coordinate system with coordinates (r,θ,z), where r is the radial distance from the central axis, θ is the angular position, and z is the position along the longitudinal axis. Due to the rotational symmetry of the cylinder and the applied pressure, the shear stress components are zero, and the state of stress at any point can be completely described by three principal stresses [6]:

1. **Radial Stress (σr):** A normal stress acting in the radial direction. It is a measure of the compressive or tensile force acting through the thickness of the cylinder wall.
2. **Circumferential or Hoop Stress (σθ):** A normal stress acting tangentially to the circumference of the cylinder. This stress is typically tensile under internal pressure and is what prevents the cylinder from bursting.
3. **Axial or Longitudinal Stress (σz):** A normal stress acting parallel to the central axis of the cylinder. This stress arises from the pressure acting on the ends of a closed cylinder or from external axial loads.

The objective of the Lamé solution is to determine the mathematical functions that describe the distribution of these three stress components, σr(r), σθ(r), and σz(r), as a function of the radial position r across the cylinder wall.[7]

1.3 Core Assumptions of Linear Elasticity Theory for the Lamé Problem

The analytical solution derived by Gabriel Lamé is a classic problem in the theory of linear elasticity. Its validity and applicability are strictly defined by a set of foundational assumptions regarding the material behavior, geometry, loading, and kinematics of the cylinder. These assumptions are not merely mathematical conveniences; they represent an idealized physical model that must be sufficiently close to the real-world conditions for the solution to be accurate.

The development of this specific solution is deeply connected to the broader theory of elasticity. The problem is a direct application of the general governing equations of continuum mechanics, which were also significantly advanced by Lamé through his definition of the fundamental elastic constants that bear his name (the Lamé parameters λ and μ).[9] The elegance of the cylinder solution lies in how a complex, three-dimensional problem is rendered analytically tractable by applying a series of physically justified assumptions. The general 3D stress-strain relations, defined by the material's elastic constants (e.g., Young's Modulus

E and Poisson's Ratio v), are combined with the equations of equilibrium. The crucial step that simplifies the governing partial differential equations into a solvable ordinary differential equation is the imposition of kinematic and symmetry constraints, namely axisymmetry and the assumption of constant longitudinal strain. Thus, the Lamé solution for cylinders is a prime example of how the general theory of elasticity can be specialized to solve important engineering problems.

The key assumptions are detailed in Table 1.

Table 1: Key Assumptions for the Lamé Solution

Category	Assumption	Implication for the Analysis
Material Behavior	**Linear Elastic**	The material obeys Hooke's Law, meaning stress is directly proportional to strain. The solution is only valid as long as the material remains within its elastic limit and does not undergo plastic deformation.[12]
	Homogeneous	The mechanical properties of the material, such as Young's Modulus (E) and Poisson's Ratio (v), are

		constant and uniform at every point throughout the cylinder's volume.[14]
	Isotropic	The material's mechanical properties are the same in all directions. This means the elastic response is independent of the orientation of the stress, and the behavior can be fully characterized by two independent elastic constants (e.g., E and v).[14]
Geometry & Loading	**Axisymmetric**	The cylinder's geometry (inner and outer radii are constant) and the applied pressure loading are symmetric about the longitudinal z-axis. This implies that the stress and displacement fields are independent of the angular position θ.[17]
	Long Cylinder	The analysis is performed on a section of the cylinder located far

		from the ends. This negates the stress-concentrating effects of end caps, flanges, or other geometric discontinuities, ensuring the stress distribution is uniform along the z-axis.[4]	
Kinematics	**Plane Sections Remain Plane**	This is a crucial kinematic assumption. It states that a plane cross-section of the cylinder that is perpendicular to the longitudinal axis before pressure is applied remains a plane after deformation.[14] This directly implies that the longitudinal strain,	ϵ_z, is constant across the entire cross-section (i.e., it is independent of the radius r).[20]

Rigorous Derivation of the Lamé Equations

The derivation of the stress distribution in a thick-walled cylinder begins from the first principles of solid mechanics: equilibrium, kinematics (strain-displacement relations), and material constitutive laws (Hooke's Law).

2.1 Equilibrium of a Differential Element in Polar Coordinates

To establish the relationship between the stress components, we consider the static equilibrium of an infinitesimal element within the cylinder wall. This element is defined in polar coordinates by a radial thickness dr, an angular width dθ, and a unit length along the z-axis. A free-body diagram of this element reveals the forces acting on its faces due to the radial stress (σr) and hoop stress (σθ).[17]

The radial stress on the inner face at radius r is σr, while on the outer face at radius r+dr, it has changed by a small amount to σr+dσr. The hoop stress σθ acts on the radial faces of the element. For the element to be in static equilibrium, the sum of all forces in the radial direction must be zero.

The forces acting radially outward are:

- Force on the outer face: $(\sigma r+d\sigma r)(r+dr)d\theta$

The forces acting radially inward are:

- Force on the inner face: $\sigma rrd\theta$
- Components of forces from hoop stress: $2(\sigma\theta dr)\sin(d\theta/2)$

For a small angle dθ, $\sin(d\theta/2)\approx d\theta/2$. The force balance equation is therefore:

$$(\sigma r+d\sigma r)(r+dr)d\theta-\sigma rrd\theta-2\sigma\theta dr2d\theta=0$$

Expanding this equation gives:

$$\sigma rrd\theta+\sigma rdrd\theta+rd\sigma rd\theta+d\sigma rdrd\theta-\sigma rrd\theta-\sigma\theta drd\theta=0$$

Dividing by dθ and neglecting the second-order term dσrdr, we are left with:

$$\sigma rdr+rd\sigma r-\sigma\theta dr=0$$

Rearranging and dividing by dr yields the fundamental differential equation of equilibrium in the radial direction [18]:

$$rdrd\sigma r+\sigma r-\sigma\theta=0$$

which can be rewritten as:

$$drd\sigma r+r\sigma r-\sigma\theta=0$$

2.2 Strain-Displacement and Constitutive Relations (Hooke's Law)

Next, we relate the strains to the physical deformation of the cylinder. Under axisymmetric loading, points on the cylinder wall displace only in the radial direction. Let u(r) be the radial displacement of a point at an initial radius r.

The hoop strain, ϵ_θ, is defined as the change in circumference divided by the original circumference. A point at radius r moves to a new radius r+u. The new circumference is $2\pi(r+u)$, and the original was $2\pi r$. Thus:

$$\epsilon_\theta = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}$$

The radial strain, ϵ_r, is the change in length per unit length in the radial direction. This is simply the gradient of the displacement field [18]:

$$\epsilon_r = \frac{du}{dr}$$

These strains are related to the stresses through the material's constitutive law. For a linear elastic, isotropic material, the generalized Hooke's Law in three dimensions relates the principal strains to the principal stresses [20]:

$$\epsilon_r = \frac{1}{E} [\sigma_r - \nu(\sigma_{\theta} + \sigma_z)]$$

$$\epsilon_{\theta} = \frac{1}{E} [\sigma_{\theta} - \nu(\sigma_r + \sigma_z)]$$

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_r + \sigma_{\theta})]$$

where E is Young's Modulus and v is Poisson's Ratio.

2.3 Formulation and Solution of the Governing Differential Equation

The derivation proceeds by combining the equilibrium, kinematic, and constitutive equations. The key is to utilize the "plane sections remain plane" assumption, which states that the longitudinal strain ϵ_z is constant across the wall thickness.[15] Furthermore, for a long cylinder away from the ends, the longitudinal stress

σ_z is also assumed to be constant.

From the third constitutive equation, if ϵ_z and σ_z are both constant, it logically follows that the term $(\sigma_r + \sigma_\theta)$ must also be constant.[20] Let us define this constant as

2A:

$$\sigma_r + \sigma_\theta = 2A$$

This allows us to express the hoop stress in terms of the radial stress and the constant A:

$$\sigma_\theta = 2A - \sigma_r$$

Now, we substitute this expression for σ_θ into the differential equation of equilibrium:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - (2A - \sigma_r)}{r} = 0$$

$$r\frac{d\sigma_r}{dr} + 2\sigma_r - 2A = 0$$

This is a first-order linear ordinary differential equation for σ_r. It can be solved by recognizing that the left side is the derivative of a product:

$$\frac{d}{dr}(r^2\sigma_r) - 2Ar = 0$$
$$\frac{d}{dr}(r^2\sigma_r) = 2Ar$$

Integrating with respect to r gives:

$$r^2\sigma_r = Ar^2 - B$$

where $-B$ is the constant of integration. Solving for σ_r yields:

$$\sigma_r = A - \frac{B}{r^2}$$

Substituting this result back into the expression for σ_θ gives:

$$\sigma_\theta = 2A - \left(A - \frac{B}{r^2}\right) = A + \frac{B}{r^2}$$

These two equations are the general form of the Lamé equations for the radial and hoop stresses in a thick-walled cylinder. They describe the stress distribution in terms of a radius-independent component A and a component B/r^2 that varies hyperbolically with the radius.[18]

2.4 The General Solution for Stresses and the Lamé Constants

The constants of integration, A and B, are determined by applying the specific boundary conditions for a given problem. For a cylinder with an inner radius r_i subjected to internal pressure p_i and an outer radius r_o subjected to external pressure p_o, the boundary conditions are defined by the radial stress at these surfaces. By convention, pressure is a compressive stress, so it is assigned a negative value.[20]

- At the inner surface: $r = r_i$, $\sigma_r = -p_i$
- At the outer surface: $r = r_o$, $\sigma_r = -p_o$

Substituting these conditions into the general equation for σ_r:

1. $-p_i = A - \frac{B}{r_i^2}$
2. $-p_o = A - \frac{B}{r_o^2}$

This is a system of two linear equations for the two unknown constants A and B. Solving this system simultaneously yields the explicit expressions for the Lamé constants in terms of the geometry and loading [18]:

$$A = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$$
$$B = \frac{(p_i - p_o) r_i^2 r_o^2}{r_o^2 - r_i^2}$$

The structure of the Lamé solution, $\sigma = A \pm B/r^2$, provides significant physical insight. The constants are not merely mathematical artifacts. By examining the expression for the longitudinal stress σ_z in a closed-end cylinder, which is found from a simple static force balance on the end caps, a deeper meaning is revealed. The net force from the internal and external pressures on the ends must be balanced by the force from the axial stress acting on the annular cross-section of the cylinder wall.[22]

$$p_i(\pi r_i^2) - p_o(\pi r_o^2) = \sigma_z(\pi r_o^2 - \pi r_i^2)$$
$$\sigma_z = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$$

This expression for σ_z is identical to the expression derived for the Lamé constant A.22 This reveals that the constant
A represents the uniform, through-thickness stress component, which is precisely the axial stress for a closed-end vessel. The term B/r^2, which decays with increasing radius, can be interpreted as the stress component required to accommodate the radial pressure gradient across the wall and satisfy equilibrium. Therefore, the complete Lamé solution can be understood as the superposition of a uniform stress field (represented by A) and a spatially varying stress field (represented by B/r^2) that together satisfy the specified pressure boundary conditions.

Analysis of a Cylinder Under Internal Pressure

We now specialize the general solution to the specific case requested by the user: a thick-walled cylinder subjected only to an internal pressure p_i.

3.1 Application of Boundary Conditions for Internal Pressure Loading

For this common and important loading case, the internal pressure is p_i and the external pressure p_o is zero (i.e., the outer surface is exposed to atmospheric pressure, which is taken as the gauge zero).[21]

The boundary conditions become:

- At $r = r_i$, $\sigma_r = -p_i$
- At $r = r_o$, $\sigma_r = 0$

Applying these to the general expressions for the Lamé constants A and B by setting po=0:

A=ro2−ri2piri2
B=ro2−ri2piri2ro2

3.2 Final Equations for Radial (σr) and Hoop (σθ) Stress

Substituting these simplified constants back into the general Lamé equations, σr=A−B/r2 and σθ=A+B/r2, gives the final equations for the stress distribution in a thick-walled cylinder under internal pressure only [4]:

Radial Stress:

$$\sigma_r(r) = \frac{p_i r_i^2}{r_o^2 - r_i^2} - \frac{p_i r_i^2 r_o^2}{(r_o^2 - r_i^2)r^2}$$Factoring out the common term gives the more standard form:$$\sigma_r(r) = \frac{p_i r_i^2}{r_o^2 - r_i^2} \left(1 - \frac{r_o^2}{r^2} \right)$$
Hoop Stress:

$$\sigma_{\theta}(r) = \frac{p_i r_i^2}{r_o^2 - r_i^2} + \frac{p_i r_i^2 r_o^2}{(r_o^2 - r_i^2)r^2}$$Factoring gives the standard form:$$\sigma_{\theta}(r) = \frac{p_i r_i^2}{r_o^2 - r_i^2} \left(1 + \frac{r_o^2}{r^2} \right)$$
These equations provide the complete stress distribution across the cylinder wall as a function of radial position r.

3.3 Distribution and Interpretation of Stress Profiles Across the Wall

A detailed analysis of these equations reveals the characteristic stress profiles in a pressurized thick-walled cylinder.

- **Radial Stress (σr):** The term (1−ro2/r2) is always zero or negative for ri≤r≤ro. Therefore, the radial stress σr is always compressive (or zero).
  - At the inner surface (r=ri): σr(ri)=ro2−ri2piri2(1−ri2ro2)=ro2−ri2piri2(ri2ri2−ro2)=−pi. The radial stress equals the applied pressure, as required by the boundary condition. This is its maximum compressive value.
  - At the outer surface (r=ro): σr(ro)=ro2−ri2piri2(1−ro2ro2)=0. The radial stress is zero at the unpressurized outer surface.
    The magnitude of the compressive radial stress decreases non-linearly from a maximum at the bore to zero at the outer surface.1

- **Hoop Stress (σθ):** The term (1+ro2/r2) is always positive. Therefore, the hoop stress σθ is always tensile.
  - At the inner surface (r=ri): σθ(ri)=ro2−ri2piri2(1+ri2ro2)=piro2−ri2ri2+ro2. This is the **maximum tensile stress** in the cylinder.
  - At the outer surface (r=ro): σθ(ro)=ro2−ri2piri2(1+ro2ro2)=piro2−ri22ri2. This is the minimum hoop stress.
    The tensile hoop stress is greatest at the inner surface and decreases non-linearly (parabolically in 1/r) towards the outer surface.1 This high concentration of hoop stress at the inner bore is the most critical feature from a design perspective.

The critical values for stress and displacement are summarized in Table 2 for quick reference.

Table 2: Summary of Stresses and Displacement at Key Locations (Internal Pressure Only)

Location	Radial Stress (σr)	Hoop Stress (σθ)	Axial Stress (σz) (Closed-End)	Radial Displacement (u) (Closed-End)
Inner Wall (r=ri)	−pi	piro2−ri2ro2+ri2	piro2−ri2ri2	Epiri(ro2−ri2ro2+ri2−v)
Outer Wall (r=ro)	0	piro2−ri22ri2	piro2−ri2ri2	E(ro2−ri2)piri2ro(2−v)

3.4 Analysis of Axial Stress (σz) for Closed-End and Open-End Conditions

The axial stress σz depends on the boundary conditions at the ends of the cylinder.

- **Closed-End Cylinder:** This is the most common case for pressure vessels. The internal pressure pi exerts a force on the end caps, which must be balanced by the axial stress in the cylinder walls. As derived in Section 2.4, this results in a uniform tensile stress across the wall given by [22]:
  σz=ro2−ri2piri2

  This is identical to the Lamé constant A for this loading case. For most geometries, this axial stress is intermediate between the maximum hoop stress and the radial stress, i.e., σθ(ri)>σz>σr(ri).

- **Open-End Cylinder (Plane Stress):** This condition applies to components like long pipelines with expansion joints, where the ends are free to move axially. In this case, there is no net axial force, and it is common to assume that the axial stress is zero throughout the wall: σz=0.[4] This is a plane stress assumption. The Lamé equations for σr and σθ are derived from radial equilibrium and are independent of the axial condition, so they remain valid for the open-end case.

3.5 Derivation and Analysis of the Radial Displacement (u)

The radial displacement u(r) of any point in the cylinder wall can be found using the constitutive relation for hoop strain:

$$\epsilon_{\theta} = \frac{u}{r} = \frac{1}{E} [\sigma_{\theta} - \nu(\sigma_r + \sigma_z)]$$

Solving for u:

u(r)=Er[σθ(r)−v(σr(r)+σz)]

Substituting the previously derived expressions for σr(r), σθ(r), and σz (for the closed-end case) into this equation:

u(r)=Er[ro2−ri2piri2(1+r2ro2)−v(ro2−ri2piri2(1−r2ro2)+ro2−ri2piri2)]

Simplifying this expression leads to the final formula for radial displacement in a closed-end cylinder under internal pressure 19:

u(r)=E(ro2−ri2)piri2[(1−v)r+(1+v)rro2]

An examination of these results reveals a fundamental principle of statically determinate elasticity problems. The equations for the stresses σr and σθ depend only on the applied pressure (pi) and the geometry (ri,ro,r). The material properties (E,v) are absent. This means that the stress state within the cylinder is independent of the material from which it is made. Two geometrically identical cylinders, one made of steel and one of aluminum, will exhibit the exact same stress distribution under the same internal pressure.

However, the equation for displacement u(r) explicitly includes Young's Modulus E and Poisson's Ratio v. This demonstrates that while the stress state is determined by equilibrium alone, the body's physical response to that stress—its deformation—is fundamentally dependent on its material properties. The aluminum cylinder, having a lower modulus of elasticity, will expand significantly more than the steel cylinder. This decoupling of the stress solution from material properties is a non-obvious but critical concept.

Comparative Analysis and Practical Application

4.1 The Thin-Walled Cylinder Approximation: A Review of Assumptions and Formulas

For context, it is useful to compare the rigorous Lamé solution with the widely used thin-walled cylinder approximation. The thin-walled theory is based on two key simplifying assumptions [5]:

1. The hoop stress ($\sigma\theta$) is assumed to be constant and uniformly distributed across the wall thickness.
2. The radial stress (σr) is considered negligible compared to the hoop stress.

Under these assumptions, a simple equilibrium balance on half of a cylinder section yields the well-known formulas for hoop stress and axial stress (for a closed-end vessel) [2]:

$\sigma\theta$=tpiri
σz=2tpiri

where t=ro−ri is the wall thickness.

4.2 Defining the Boundary: Quantitative Criteria for Thick-Walled Analysis

The practical question for an engineer is when the simplified thin-walled theory is acceptable and when the more complex Lamé solution is necessary. The decision is typically based on the cylinder's geometry, but a more nuanced approach also considers the loading conditions.

The most common criterion is a geometric one. A cylinder is generally considered "thick-walled," requiring the Lamé analysis, when the ratio of its inner diameter to its wall thickness (di/t) is less than 20, or equivalently, when its inner radius to wall thickness ratio (ri/t) is less than 10.[2] Some sources use a slightly more conservative threshold of

di/t<15.[3]

However, this geometric rule is not a rigid physical law but rather an engineering guideline based on an acceptable level of error. For a cylinder with $r_i/t>10$, the actual variation in hoop stress across the wall is typically less than 5%.[16] For many applications, this level of inaccuracy is acceptable, and the thin-walled formulas suffice. For high-precision or safety-critical applications, a thick-walled analysis may be warranted even for geometries that are borderline "thin."

Furthermore, a stress-based criterion can also be applied. Some design codes classify a cylinder as thick-walled if the internal pressure p_i is greater than one-sixth of the material's allowable stress ($\sigma_{allowable}$).[4] This acknowledges that even if a cylinder is geometrically thin, very high pressures can make the radial stress (which is approximately

$-p_i$ at the bore) significant relative to the material's strength, violating a key assumption of the thin-walled model. The choice of analytical model is therefore a matter of engineering judgment, considering the geometry, loading intensity, material properties, and the required fidelity of the results.

Table 3 provides a direct comparison of the two theories.

Table 3: Comparison of Thick-Walled vs. Thin-Walled Cylinder Theories

Attribute	Thin-Walled Theory	Thick-Walled Theory (Lamé Solution)
Geometric Criterion	Inner Radius / Thickness (r_i/t) > 10	Inner Radius / Thickness (r_i/t) < 10
Radial Stress (σ_r)	Assumed to be negligible ($\sigma_r \approx 0$).	Significant and compressive. Varies from $-p_i$ at the inner wall to zero at the outer wall.
Hoop Stress (σ_θ)	Assumed to be constant across the wall thickness. Calculated as $\sigma_\theta = p_i r_i / t$.	Varies non-linearly across the thickness. Maximum at the inner wall and minimum at the outer wall.
Stress Distribution	Uniform stress distribution is assumed.	Non-uniform stress distribution is calculated.
Applicability	Low-pressure applications,	High-pressure applications,

	boilers, storage tanks where the wall is thin relative to the radius.[2]	hydraulic cylinders, gun barrels, where wall thickness is significant.[2] Can handle both internal and external pressures.

4.3 A Quantitative Comparison of Stress Predictions

To illustrate the difference, consider a cylinder with an inner radius $r_i=100$ mm and an outer radius $r_o=110$ mm, which gives a thickness $t=10$ mm. The ratio $r_i/t=10$, placing it on the boundary between thin and thick. Let the internal pressure be $p_i=50$ MPa.

- Using Thin-Walled Theory:
  The predicted hoop stress is constant:

  $$\sigma_\theta = \frac{p_i r_i}{t} = \frac{50\text{ MPa} \times 100\text{ mm}}{10\text{ mm}} = 500\text{ MPa}$$
- Using Thick-Walled (Lamé) Theory:
  The maximum hoop stress occurs at the inner wall ($r=r_i$):

  $$\sigma_{\theta,max} = p_i \frac{r_o^2+r_i^2}{r_o^2-r_i^2} = 50\text{ MPa} \times \frac{(110^2+100^2)}{(110^2-100^2)} = 50 \times \frac{22100}{2100} \approx 526.2\text{ MPa}$$

In this borderline case, the thin-walled theory underestimates the maximum stress by over 5%. For a cylinder with $r_i/t=5$ (e.g., $r_o=120$ mm), the Lamé solution gives a maximum hoop stress of 327 MPa, while the thin-walled formula gives 250 MPa, an underestimation of over 23%. This demonstrates the critical importance of using the correct theory for accurate and safe design.

4.4 Failure Theories for Pressurized Cylinders

The ultimate purpose of stress analysis is to prevent failure. For ductile materials commonly used in pressure vessels, failure is typically governed by the onset of yielding, which is predicted by failure theories based on the state of stress. The **Maximum Shear Stress Theory (Tresca Criterion)** is particularly well-suited for this problem. It states that yielding begins when the maximum shear stress in the material reaches the shear stress at yield in a simple tension test.

The principal stresses in the cylinder are $\sigma_1=\sigma_\theta$, $\sigma_2=\sigma_z$, and $\sigma_3=\sigma_r$. Since σ_θ is the largest tensile stress and σr is compressive (negative), the maximum shear stress at any point in the wall is given by [6]:

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{\sigma_{\theta} - \sigma_r}{2}$$Substituting the Lamé equations for σ_{θ} and σ_r:$$\tau_{max}(r) = \frac{1}{2} \left = \frac{B}{r^2}$$For the case of internal pressure only, this becomes:$$\tau_{max}(r) = \frac{p_i r_i^2 r_o^2}{(r_o^2 - r_i^2)r^2}$$
This result leads to a powerful conclusion. The maximum shear stress is inversely proportional to r2. Therefore, its value is mathematically guaranteed to be at its absolute maximum at the smallest possible radius, which is the inner radius, r=ri. This means that for any thick-walled cylinder made of a ductile material and subjected to internal pressure, material yielding will *always* initiate at the inner surface (the bore). This analytical prediction aligns perfectly with experimental observations, which show that fatigue cracks and ruptures in pressurized pipes typically start from the inside.[6] The Lamé solution provides the direct theoretical link between the applied load and this specific, observable failure mechanism.

Conclusion: Synthesis and Advanced Perspectives

5.1 Summary of the Lamé Solution and its Implications

The Lamé solution provides a complete, analytical description of the elastic stress and displacement fields in a thick-walled cylinder subjected to pressure. It is a cornerstone of mechanical and structural engineering, derived directly from the fundamental principles of equilibrium and linear elasticity. The key findings of this analysis are:

- The radial stress (σr) is compressive, varying from a maximum magnitude equal to the internal pressure at the inner bore to zero at the unpressurized outer surface.
- The hoop stress (σθ) is tensile, reaching its maximum value at the inner bore and decreasing non-linearly towards the outer surface.
- This concentration of stress at the inner surface makes it the most critical location for design against failure. The maximum shear stress is always located at the inner bore, dictating that yielding will initiate at this point.
- The stress distribution in a statically determinate cylinder is independent of material properties, whereas the resulting displacement is fundamentally dependent on the material's Young's Modulus and Poisson's Ratio.

The analysis underscores the inadequacy of thin-walled approximations for components where the wall thickness is significant relative to the radius, highlighting the necessity of applying the more rigorous Lamé equations for the design of high-pressure systems.

5.2 Limitations of the Elastic Solution

It is crucial to recognize the boundaries of the Lamé solution. The entire derivation is predicated on the assumption of linear elastic material behavior. If the internal pressure is high enough to cause the maximum effective stress (e.g., based on the Tresca or von Mises criterion) at the inner bore to exceed the material's yield strength, the material will begin to deform plastically.[4] Once yielding occurs, the linear stress-strain relationship is no longer valid, and the Lamé equations cease to accurately describe the stress state. In such cases, a more complex elastic-plastic analysis is required to determine the stress distribution and the extent of the plastic zone.

5.3 Introduction to Advanced Topics: Compound Cylinders and Autofrettage

The inherent stress concentration at the inner bore of a thick-walled cylinder limits its pressure-carrying capacity. Engineers have developed advanced techniques to overcome this limitation by inducing a favorable residual stress state in the cylinder before it is put into service. These methods are direct, practical applications that build upon the foundational understanding of the Lamé stress distribution.

- **Compound Cylinders (Shrink Fits):** This technique involves manufacturing the cylinder from two or more concentric shells. The outer cylinder is heated or the inner cylinder is cooled, and they are fitted together. Upon returning to a uniform temperature, the interference in their dimensions creates a large contact pressure at the interface. This "shrink-fit" pressure places the inner cylinder into a state of residual compression and the outer cylinder into residual tension.[21] When internal working pressure is subsequently applied, the initial compressive hoop stress at the inner bore must first be overcome before the material goes into tension. This effectively increases the pressure capacity of the composite cylinder.[6]
- **Autofrettage:** This process is applied to a single cylinder. It involves subjecting the cylinder to an extremely high internal pressure, sufficient to cause yielding and plastic flow in the material near the inner bore, while the outer region remains elastic. When this high pressure is released, the elastic outer region attempts to return to its original size, but is prevented from doing so by the permanently deformed inner region. This results in

the outer region placing the inner region into a state of high residual compressive stress.[6] Much like a shrink-fit, this residual compression significantly enhances the cylinder's resistance to fatigue and increases its elastic operating pressure range.

Both techniques demonstrate how a thorough understanding of the elastic stress distribution provided by the Lamé solution is the essential first step toward designing even more advanced and robust high-pressure containment systems.

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